Transfer matrices and conductivity in two- and three-dimensional systems. II. Application to localised and delocalised systems

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Transfer matrices and conductivity in two- and three-dimensional systems: II. Application to localised and delocalised systems

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Abstract. In the previous paper a formalism was established for the calculation of intensities reflected from a disordered system, and hence for the transmitted intensities via the unitarity relationship. Here we show how the method can be applied to calculation of the conductivity of a three-dimensional (3D) system in the limiting case of weak disorder. First the one-dimensional (1D) situation is discussed and canonical models developed for the classical diffusive case and for the quantum localised case. Our 3D theory can then be mapped onto the classical 1D case in the limit of weak disorder, and onto the quantum 1D case in the limit that all laterial hopping is eliminated and we have a collection of independent 1D systems. Thus our theory has the power and flexibility to describe both localised and delocalised systems, a unique advantage in discussing more complex effects.

1. Introduction

In paper I of the series (Pendry 1990) I introduced a formalism to calculate reflection coefficients of 3D disordered quantum systems. From the unitarity relationship for a purely elastic system this also enables us to calculate the transmitted intensities

$$\boldsymbol{r}\boldsymbol{r}^{+} + \boldsymbol{t}\boldsymbol{t}^{+} = \mathbf{1} \tag{1}$$

and hence the conductance of a system with dimensions $L \times L$,

$$G = \operatorname{tr} \boldsymbol{t}\boldsymbol{t}^{+}.$$

In this paper the formalism is applied to the case of weak disorder where we expect delocalised behaviour, the conductance obeying Ohm's law,

$$G_{L_z} \simeq \text{const}/L_z. \tag{2}$$

There are several objectives in the paper. The first is to demonstrate that our formalism is tractable and can be applied to give sensible results in situations that we can understand from other methods. The second is to show for the first time that a transfer matrix method can correctly describe the delocalised regime.

This latter point is not a trivial one because transfer matrices introduce the thickness of the system by exponentiation, so that we have a formula for the conductance of the form

$$G_{L_z} \simeq \langle a|\mathbf{X}^{L_z}|b\rangle \tag{3}$$

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where **X** is a transfer matrix. Thus transfer matrices describe in a natural way the exponential decay of conductance with thickness characteristic of a localised regime. Some subtle juggling of the spectrum has to take place before delocalisation can be accounted for.

General background papers to the localisation problem can be found in the proceedings of the 1984 Braunschweig conference (Kramer *et al* 1985). References to the current approach are to be found in paper I.

Our hope is that, once we have shown how to describe this delocalised behaviour, the natural affinity of the transfer matrix for localisation will put within our grasp a microscopic quantitative description of the Anderson transition. There is considerable controversy over the critical exponents at the transition, those exploiting scaling (Abrahams *et al* 1979) or renormalisation-group methods (Wegner 1976) suggesting an exponent of 1 in 3D, and other making numerical simulations (Schreiber *et al* 1989) reporting $\frac{3}{2}$ for systems of the same general character. A model of the transition that could be solved analytically would be invaluable in resolving this debate.

As I have indicated, the solution of the delocalised problem by transfer matrix methods can be expected to be non-trivial. Hence the paper begins with a treatment of the very simple problem of diffusion of a classical fluid in 1D. This provides the canonical form of a delocalised transfer matrix and the 3D quantum problem is then solved by transforming the 3D transfer matrix into the 1D form. The classical 1D transfer matrix and its eigenvalues are contrasted with the 1D quantum case, which shows only localised behaviour.

2. Transport in one dimension—a classical model

In a transfer matrix approach, transmission of electrons through a layer of disordered material is represented by exponentiation of a set of eigenvalues. Thus the length dependence of the conductance is naturally exponential: something special has to happen in the mathematics to obtain Ohm's law, $G_{L_z} = \text{const}/L_z$, from a set of exponentials. However, the weak scattering case, where we known that Ohm's law holds, is well understood from other approaches. In this section we go through the exercise of solving the simplest problem that exhibits Ohm's law using the methods of symmetrised transfer matrices.

Let us imagine a continuous fluid flowing through a pipe that has a series of obstacles in it. At the nth obstacle a fraction r_n of the fluid is reflected, and a fraction t_n transmitted. We assume current conservation, i.e.

$$t_n + r_n = 1. \tag{4}$$

In order to simplify the model as far as possible we assume that

$$r_n = r \qquad t_n = t. \tag{5}$$

It might be imagined that this simple model, devoid even of fluctuations, is completely trivial. It is not trivial, as we shall show, and it will serve as the archetypal diffusive system onto which we shall ultimately map the quantum problem.

The transfer matrix can be constructed in the usual way. Between the $(n-1)$th and nth layers there is a current a_n^+ flowing to the right, and a current a_n^- flowing to the left.

The transmission and reflection coefficients of the nth layer relate amplitudes as follows:

$$a_{n+1}^{+} = ta_n^{+} + ra_{n+1}^{-} \tag{6a}$$

$$a_n^{-} = ra_n^{+} + ta_{n+1}^{-} \tag{6b}$$

or in matrix form

$$\begin{bmatrix} 1 & -r \\ 0 & -t \end{bmatrix} \begin{bmatrix} a_{n+1}^{+} \\ a_{n+1}^{-} \end{bmatrix} = \begin{bmatrix} t & 0 \\ r & -1 \end{bmatrix} \begin{bmatrix} a_n^{+} \\ a_n^{-} \end{bmatrix}. \tag{7}$$

It follows by matrix inversion that

$$\boldsymbol{a}_{n+1} = \mathbf{T}\boldsymbol{a}_n \tag{8}$$

where

$$\mathbf{T} = \begin{bmatrix} t - r^2t^{-1} & rt^{-1} \\ -rt^{-1} & t^{-1} \end{bmatrix}. \tag{9}$$

Re-expressing t in terms of r we obtain

$$\mathbf{T} = \begin{bmatrix} (1-2r)/(1-r) & r/(1-r) \\ -r/(1-r) & 1/(1-r) \end{bmatrix}. \tag{10}$$

We can use $\mathbf{T}$ to calculate the transmission coefficient of L_z layers, t_{L_z}. From the properties of transfer matrices,

$$\mathbf{T}_{L_z} = \prod_{n=1}^{L_z} \mathbf{T}_n \tag{11}$$

where $\mathbf{T}_{L_z}$ is the transfer matrix for a stack of L_z layers. From the definition of the transfer matrix it follows that

$$t_{L_z}^{-1} = (1 - r_{L_z})^{-1} = (\mathbf{T}_{L_z})_{22}. \tag{12}$$

Provided that we can find the eigenvalues of $\mathbf{T}$ we can obtain

$$t_{L_z}^{-1} = \sum_j e_j^{L_z} \boldsymbol{v}_{j2}^{\mathrm{r}} \boldsymbol{v}_{j2}^{\mathrm{l}} \tag{13}$$

where $\boldsymbol{v}^{\mathrm{l}}$ and $\boldsymbol{v}^{\mathrm{r}}$ are the left and right eigenvectors of $\mathbf{T}$. We can exploit the fact that there are no fluctuations in the system to invert $1/t$, i.e.

$$t = 1 \Big/ \left(\sum_j e_j^{L_z} \boldsymbol{v}_{j2}^{\mathrm{r}} \boldsymbol{v}_{j2}^{\mathrm{l}} \right). \tag{14}$$

We now arrive at the interesting fact that $\mathbf{T}$ is a matrix with degenerate eigenvalues and parallel eigenvectors. This curious fact is crucial to removing the exponential dependence on l_z. First we must remove the singularity that this degeneracy imposes on the system. We do this by defining a new matrix

$$\mathbf{T}(\alpha) = \begin{bmatrix} (1-2r)/(1-r) & (r+\alpha)/(1-r) \\ (-r+\alpha)/(1-r) & 1/(1-r) \end{bmatrix} \tag{15}$$

and taking the limit $\alpha \to 0$. The eigenvalues of $\mathbf{T}(\alpha)$ are

$$e_{\pm} = 1 \pm \alpha \simeq \exp(\pm\alpha)$$

with the corresponding eigenvectors

$$\boldsymbol{v}^{\mathrm{r}}_{\pm} = \begin{bmatrix} 1 \\ \{r - \alpha(1-r)\}/\{r \mp \alpha(1-r)\} \end{bmatrix} \tag{16}$$

$$\boldsymbol{v}^{\mathrm{l}}_{\pm} = [1, \quad -\{r + \alpha(1-r)\}/\{r \mp \alpha(1-r)\}]. \tag{17}$$

Evaluating equation (13) and assuming α small gives

$$t^{-1}_{L_z} \underset{\lim \alpha \to 0}{=} \frac{r + \alpha(1-r)}{2\alpha(1-r)} \exp(\alpha L_z) - \frac{r - \alpha(1-r)}{2\alpha(1-r)} \exp(-\alpha L_z)$$

$$= 1 + \frac{r}{(1-r)} L_z. \tag{18}$$

Assuming that the conductance

$$G_{L_z} = \text{const}' \, t \tag{19}$$

then we retrieve conventional Ohm's law behaviour from our model,

$$G_{L_z} = \{1 + [r/(1-r)] L_z\}^{-1} \simeq t/(rL_z). \tag{20}$$

Again, since this model contains no statistics, we can calculate any power of G_{L_z} simply by raising G_{L_z} to the appropriate power. This result would not be true if we had calculated an average for G_{L_z}.

Now let us formulate the problem in a more sophisticated fashion, one that bears closer resemblance to our genralised transfer matrix approach used in the quantum case (see paper I).

Matrix elements of the symmetrised Nth-order transfer matrix can be written as

$$\mathbf{X}^N_{ij} = \sum_{p=0}^{\min(i,j)} \frac{i!}{p!(i-p)!} \frac{(N-i)!}{(j-p)!\,(N+p-i-j)!} \mathbf{T}^{p}_{11} \mathbf{T}^{i-p}_{12} \mathbf{T}^{j-p}_{21} \mathbf{T}^{N-i-j+p}_{22} \tag{21}$$

where the subscripts i and j run from 0 to ∞. The general form of $\mathbf{X}$ is

$$\mathbf{X}^N_{L_z} = \begin{bmatrix} t^{-N}_{L_z} & -Nr_{L_z} t^{-N}_{L_z} & [N(N-1)/2] r^2_{L_z} t^{-N}_{L_z} & \dots \\ r_{L_z} t^{-N}_{L_z} & t^{2-N}_{L_z} - Nr^2_{L_z} t^{-N}_{L_z} & \dots & \dots \\ r^2_{L_z} t^{-N}_{L_z} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{22}$$

Hence we can find any power of r_{L_z} from this matrix,

$$r^i_{L_z} = \lim_{N \to 0} t^N_{L_z} (\mathbf{X}^N_{L_z})_{i0} \tag{23}$$

where $\mathbf{X}^N_{L_z}$ is found in the usual way,

$$\mathbf{X}^N_{L_z} = \prod_{n=1}^{L_z} \mathbf{X}^N_n \tag{24}$$

and if we make the weak scattering approximation so that we can neglect powers of r higher than the first, then $\mathbf{X}^N$ has the form of a tridiagonal matrix

$$\mathbf{X}^N = \begin{bmatrix} 1+Nr & -Nr & 0 & \dots & \dots & \dots & \dots \\ r & 1+(N-2)r & -(N-1)r & 0 & \dots & \dots & \dots \\ 0 & 2r & 1+(N-4)r & -(N-2)r & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \mathrm{i}r & 1+(N-2\mathrm{i})r & -(N-\mathrm{i})r & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}. \tag{25}$$

In this case we have already calculated the result,

$$\overline{r_{L_z}^N} = r_{L_z}^N = [\![1 - \{1 + [r/(1-r)]L_z\}^{-1}]\!]^N. \tag{26}$$

We note that diagonalisation of $\mathbf{X}$ is likely to be a non-trivial exercise because the diagonal elements are degenerate to zeroth order in r, thus precluding perturbative solution. This difficulty has been circumvented because the quantities in which we are interested can be calculated by other means in this instance. Even if we are faced with a more intractable matrix, provided that we can reduce it to the canonical form (25), the solution can be obtained immediately from the above.

3. Transport in one dimension—the quantum case

It is an interesting exercise to contrast the transfer matrix that describes classical diffusive behaviour in 1D with the quantum version that describes only localised behaviour.

The quantum transfer matrix becomes

$$\mathbf{TQ} = \begin{bmatrix} t - r^2 t^{-1} & rt^{-1} \\ -rt^{-1} & t^{-1} \end{bmatrix} = \begin{bmatrix} t^{*-1} & -r^* t^{*-1} \\ -rt^{-1} & t^{-1} \end{bmatrix}. \tag{27}$$

We have used time-reversal invariance to reduce the matrix to a simplified form. In this expression the t and r refer to amplitudes not to intensities. Again we construct a generalised transfer matrix

$$\mathbf{XQ}_{ij}^N = \sum_{p=0}^{\min(i,j)} \frac{i!}{p!(i-p)!} \frac{(N-i)!}{(j-p)!(N+p-i-j)!} \times \mathbf{TQ}_{11}^{p}\, \mathbf{TQ}_{12}^{i-p}\, \mathbf{TQ}_{21}^{j-p}\, \mathbf{TQ}_{22}^{N-i-j+p} \tag{28}$$

and for a system of length L_z this matrix takes the form

$$\mathbf{XQ}_{L_z}^N = \begin{bmatrix} t_{L_z}^{-N} & -Nr_{L_z}t_{L_z}^{-N} & [N(N-1)/2]r_{L_z}^2 t_{L_z}^{-N} & \dots \\ -r_{L_z}t_{L_z}^{-N} & t_{L_z}^{2-N} - Nr_{L_z}^2 t_{L_z}^{-N} & \dots & \dots \\ r_{L_z}^2 t_{L_z}^{-N} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{29}$$

Taking the complex conjugate we get

$$\mathbf{XQ}^{*N}_{L_z} = \begin{bmatrix} t^{*-N}_{L_z} & -Nr^{*}_{L_z}t^{*-N}_{L_z} & [N(N-1)/2]r^{*2}_{L_z}t^{*-N}_{L_z} & \dots \\ -r^{*}_{L_z}t^{*-N}_{L_z} & t^{*2-N}_{L_z} - Nr^{*2}_{L_z}t^{*-N}_{L_z} & \dots & \dots \\ r^{*2}_{L_z}t^{*-N}_{L_z} & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{bmatrix}. \tag{30}$$

Finally taking the direct product of these matrices (see Pendry and Kirkman 1984) and arranging the subscripts in such an order that the resulting matrix has the form

$$\mathbf{XQ}^{*N}_{L_z} \otimes \mathbf{XQ}^{N}_{L_z} = \begin{bmatrix} t^{*-N}_{L_z}t^{-N}_{L_z} & N^2 r^{*}_{L_z}t^{*-N}_{L_z}r_{L_z}t^{-N}_{L_z} & \dots \\ r^{*}_{L_z}t^{*-N}_{L_z}r_{L_z}t^{-N}_{L_z} & (t^{*2-N}_{L_z} - Nr^{*2}_{L_z}t^{*-N}_{L_z})(t^{2-N}_{L_z} - Nr^{2}_{L_z}t^{-N}_{L_z}) & \dots \\ r^{*2}_{L_z}t^{*-N}_{L_z}r^{2}_{L_z}t^{-N}_{L_z} & \dots & \dots \\ \dots & \dots & \dots \end{bmatrix} \tag{31}$$

we can obtain powers of $|r_{L_z}|^{2i}$ from

$$|r_{L_z}|^{2i} = \lim_{N\to 0} (\mathbf{XQ}^{*N}_{L_z} \otimes \mathbf{XQ}^{N}_{L_z})_{i0}. \tag{32}$$

As before we wish to use the fundamental formula

$$\mathbf{XQ}^{*N}_{L_z} \otimes \mathbf{XQ}^{N}_{L_z} = \prod_{n=1}^{L_z} \mathbf{XQ}^{*N}_{n} \otimes \mathbf{XQ}^{N}_{n}. \tag{33}$$

We take a simple model of a weakly disordered 1D chain in which the atomic energies are disordered by a small amount δ_n, so that

$$\begin{aligned} 1/t_n &= (1 - \mathrm{i}\delta'_n)\exp(\mathrm{i}k) \\ r_n/t_n &= -\mathrm{i}\delta'_n \exp(\mathrm{i}k) \\ \delta'_n &= \delta_n/2\sin(k) \end{aligned} \tag{34}$$

where k is the wavevector for an ordered system. Using these expressions we can calculate

$$\lim_{N\to 0} \overline{\mathbf{XQ}^{*N}_{n} \otimes \mathbf{XQ}^{N}_{n}}$$

$$= \begin{bmatrix} 1+\delta'^2 N & \delta'^2 N^2 & 0 & \dots & \dots & \dots & \dots \\ \delta'^2 & 1-2\delta'^2 & \delta'^2 & 0 & \dots & \dots & \dots \\ 0 & 4\delta'^2 & 1-8\delta'^2 & 4\delta'^2 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & j^2\delta'^2 & 1-2j^2\delta'^2 & j^2\delta'^2 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{35}$$

and from our formulae for the reflection coefficient and its relationship to the transmission coefficient the result for $|t|^2$ and its moments follows (Pendry and Kirkman 1984):

$$|t|^2 = \{1/[2(\pi L_z \delta'^2)^{3/2}]\} \exp(-L_z \delta'^2/4) \tag{36}$$

$$(|t|^2)^N = |t|^2[\Gamma^2(N - \tfrac{1}{2})\Gamma^2(1)]/[\Gamma^2(\tfrac{1}{2})\Gamma^2(N)] \tag{37}$$

where Γ is the gamma function. The first of these formulae depends on the assumption that δ^2 is small; the second is a general result for 1D systems valid in the limit of large L_z.

Of course our arrangement of the subscripts hides other terms of this matrix in which $(\mathbf{XQ}_n)_{jj}$ is not paired with $(\mathbf{XQ}_n^*)_{jj}$ along the diagonal. However, these terms are not degenerate with the ones shown above and therefore can be neglected provided that δ^2 is small.

One purpose of deriving (35) is to compare and contrast it with the classical diffusive case. Note the similarities: both (25) and (35) describe a tridiagonal matrix; neither contains any phase information. Yet the differences are crucial: in the quantum case (35) contains terms increasing down the matrix as j^2; in contrast the classical case has terms increasing only as j. We know that, in the quantum case, localisation occurs and fluctuations are extremely strong, whereas, in the particularly simple classical model we have chosen, there are no fluctuations. It is the rapid increase of matrix elements in the quantum case that generates these fluctuations.

I shall state without proof several interesting results concerning the matrices (25) and (35):

(i) both have a continuous spectrum of eigenvalues;
(ii) in the quantum case the upper limit on the spectrum is $1 - \delta'^2/4$;
(iii) in the classical case the upper limit of the spectrum is 1;
(iv) if we try to calculate eigenvalues numerically by truncating the infinite matrices to dimensions $m \times m$, then the eigenvalues are in error due to the truncation by order $1/\ln^2 m$ in the quantum case; and
(v) in the classical case the error is of order $1/m$.

4. Model for a three-dimensional disordered system

In paper I we considered the 3D quantum case and showed that the 3D transfer matrix has the form

$$\mathbf{T} = \begin{bmatrix} \mathbf{T}'^{++}_{kk'} & \mathbf{T}'^{+-}_{kk'} \\ \mathbf{T}'^{-+}_{kk'} & \mathbf{T}'^{--}_{kk'} \end{bmatrix}. \tag{38}$$

Now we consider a specific model for $\mathbf{T}$. The system is taken to consist of L_z planes of atoms stacked along the z direction, each plane containing a square $L \times L$ array of atoms. A single tight-binding level is associated with each atom, energy E_{nr}, where $\boldsymbol{r} = (x, y)$ labels the atoms in the plane and n labels the plane. The atoms are coupled together by hopping integrals, V in the plane and V' out of the plane. A detailed description of the model is given by Pendry and Kirkman (1984). Here we shall work in a basis set consisting of the Bloch waves of the ordered system, wavevectors given by

$$\exp(\mathrm{i}\boldsymbol{K}_k) = [\Gamma_k + \mathrm{i}(-\Gamma_k^2 + 4)^{1/2}]/2 \tag{39}$$

$$\Gamma_k = \overline{E_{nr}/V'} - 2(V/V')\cos(2\pi \boldsymbol{k}_x) - 2(V/V')\cos(2\pi \boldsymbol{k}_y). \tag{40}$$

In this basis the four quadrants of $\mathbf{T}$ are given by

$$\mathbf{T}'^{++}_{kk'} = \exp(-\mathrm{i}K_{k'})\delta_{kk'} + A_k^* \exp(-\mathrm{i}K_{k'})D^*_{-k+k'} \tag{41a}$$

$$\mathbf{T}'^{+-}_{kk'} = A_k^* \exp(+\mathrm{i}K_{k'})D^*_{-k+k'} \tag{41b}$$

$$\mathbf{T}'^{-+}_{kk'} = A_k \exp(-\mathrm{i}K_{k'})D_{k-k'} \tag{41c}$$

$$\mathbf{T}'^{--}_{kk'} = \exp(+\mathrm{i}K_{k'})\delta_{kk'} + A_k \exp(+\mathrm{i}K_{k'})D_{k-k'} \tag{41d}$$

where

$$A_k = 1/[2\mathrm{i}\sin(K_k)] \tag{42}$$

$$\overline{D_{k-k'}D^*_{k-k'+q}} = \begin{cases} 0 & q \neq 0 \\ \delta^2/L^2 & q = 0 \end{cases} \tag{43}$$

and

$$\delta^2 = \overline{[E^2_{nr}} - \overline{(E_{nr})^2}]/V'^2. \tag{44}$$

We shall further restrict the model to the case

$$V' \gg V \tag{45}$$

which simply means that electrons are much more mobile in the z direction than they are within the planes. If we consider energies not too far from the centre of the band, this implies that all the $\boldsymbol{K}$ are real, and that

$$A_k \simeq 1/(2\mathrm{i}) \tag{46}$$

simplifying the algebra somewhat without losing the essentials of the problem.

The next task is to construct the $\mathbf{Y}$ matrix, which is an array of determinants formed by choosing all possible selections of L^2 rows and L^2 columns from the $2L^2 \times 2L^2$ matrix $\mathbf{T}$. To this end we arrange the determinants into hierarchies characterised by the number of the rows or columns chosen from the second set of L^2 rows or columns of $\mathbf{T}$. We shall assume that

$$\delta^2 \ll 1 \tag{47}$$

and neglect terms smaller than δ^2. The first choice is zero rows and columns from the second half,

$$\mathbf{Y}(0;0) = \det(\mathbf{T}^{++}). \tag{48}$$

The second choice involves omitting row $\boldsymbol{k}_1$ from the first set and choosing instead row $\boldsymbol{k}_2$ from the second set,

$$\mathbf{Y}(\boldsymbol{k}_1\boldsymbol{k}_2;0) = \det(\mathbf{T}^{++})D_{k_2-k_1}/2\mathrm{i}. \tag{49}$$

In the third choice we omit column $\boldsymbol{k}'_1$ from the first set and choose instead column $\boldsymbol{k}'_2$ from the second set,

$$\mathbf{Y}(0;\boldsymbol{k}_1;\boldsymbol{k}'_2) = -\det(\mathbf{T}^{++})\exp(\mathrm{i}K_{k'_1} + \mathrm{i}K_{k'_2})D^*_{-k'_1+k'_2}/2\mathrm{i}. \tag{50}$$

The fourth choice omits row $\boldsymbol{k}_1$ from the first set and chooses row $\boldsymbol{k}_2$ from the second set, omits column $\boldsymbol{k}'_1$ from the first set and chooses column $\boldsymbol{k}'_2$ from the second set,

$$\mathbf{Y}(\boldsymbol{k}_1,\boldsymbol{k}_2;\boldsymbol{k}_1;\boldsymbol{k}'_2) = \det(\mathbf{T}^{++})\exp(\mathrm{i}K_{k'_1} + \mathrm{i}K_{k'_2})\,[(1-\delta^2/2)\delta_{k_1k'_1}\delta_{k_2k'_2}$$
$$+ \delta_{k_1k'_1}D_{k_2-k'_2}/2\mathrm{i} + \delta_{k_2k'_2}D_{-k'_1+k_1}/2\mathrm{i}]. \tag{51}$$

This is the minimum number of choices needed to get a consistent picture of the transfer matrix for reflectivities. There are other contributions to the **Y** matrix, which involve multiple choices from the second set of rows and columns, but we shall not include them in our calculations.

Next we must construct the **X** matrix. The lower-order terms are given by

$$\mathbf{X}(0;0) = \mathbf{Y}^N(0;0) \tag{52}$$

$$\mathbf{X}(0;k_1',k_2') = N\mathbf{Y}^N(0;0)D^*_{-k_1'+k_2'}\exp(\mathrm{i}K_{k_1'} + \mathrm{i}K_{k_2'}) \tag{53}$$

$$\mathbf{X}(k_1',k_2';0) = \mathbf{Y}^N(0;0)D_{k_2-k_1} \tag{54}$$

$$\begin{aligned}\mathbf{X}(k_1,k_2;k_1';k_2') &= \exp(\mathrm{i}K_{k_1'} + \mathrm{i}K_{k_2'})[\mathbf{Y}^{N-1}(0;0)\mathbf{Y}(k_1,k_2;k_1',k_2')\\ &\quad + (N-1)\,\mathbf{Y}^{N-2}(0;0)\mathbf{Y}(0;k_1',k_2')\mathbf{Y}(k_1k_2;0)]\\ &\simeq \exp(\mathrm{i}K_{k_1'} + \mathrm{i}K_{k_2'})\mathbf{Y}^N(0;0)\{(1-\delta^2/2)\delta_{k_1'k_1}\delta_{k_2'k_2}\\ &\quad + \delta_{k_1k_1'}D_{k_2-k_2'}/2\mathrm{i} + \delta_{k_2k_2'}D^*_{-k_1'+k_1}/2\mathrm{i}\\ &\quad + [(N-1)\delta^2/(4L^2)]\delta_{(k_1'-k_2',k_1-k_2)}\}.\end{aligned} \tag{55}$$

The nth-order diagonal term has the form

$$\begin{aligned}\mathbf{X}(n,k_1k_2,k_3k_4,\ldots;n,k_1'k_2',k_3'k_4',\ldots) &\simeq \mathbf{Y}^N(0;0)\exp(\mathrm{i}K_{k_1'} + \mathrm{i}K_{k_2'},\\ &\quad + \ldots + \mathrm{i}K_{k_{2n-1}'} + \mathrm{i}K_{k_{2n}'})\{(1-n\delta^2/2)\delta_{k_1'k_1}\delta_{k_2'k_2}\ldots\delta_{k_{2n-1}'k_{2n-1}}\delta_{k_{2n}'k_{2n}}\\ &\quad + D_{k_2-k_2'}\delta_{k_1'k_1}\delta_{k_3'k_3}\ldots\delta_{k_{2n-1}'k_{2n-1}}\delta_{k_{2n}'k_{2n}}\\ &\quad + D^*_{-k_1'+k_1}\delta_{k_2'k_2}\delta_{k_3'k_3}\ldots\delta_{k_{2n-1}'k_{2n-1}}\delta_{k_{2n}'k_{2n}}\\ &\quad + [(N-n)\delta^2/(4L^2)]\delta_{k_1'-k_2',k_1-k_2}\delta_{k_3'k_3}\ldots\delta_{k_{2n-1}'k_{2n-1}}\delta_{k_{2n}'k_{2n}}\}\end{aligned} \tag{56}$$

and the off-diagonals, assuming single occupancy of all the levels, have the form

$$\begin{aligned}&\mathbf{X}(n,k_1k_2,k_3k_4,\ldots;n-1,k_1'k_2',k_3'k_4',\ldots)\\ &\quad\simeq \mathbf{Y}^{N-1}_{(0;0)}\mathbf{Y}(k_{2n},k_{2n+1};0)\exp(\mathrm{i}K_{k_1'} + \mathrm{i}K_{k_2'} + \ldots + \mathrm{i}K_{k_{2n-3}'} + \mathrm{i}K_{k_{2n-2}'})\\ &\qquad\times \delta_{k_1'k_1}\delta_{k_2'k_2}\ldots\delta_{k_{2n-3}'k_{2n-3}}\delta_{k_{2n-2}'k_{2n-2}}\\ &\quad= \mathbf{Y}^{N-1}_{(0;0)}D_{k_{2n}-k_{2n+1}}\exp(\mathrm{i}K_{k_1'} + \mathrm{i}K_{k_2'}, + \ldots\mathrm{i}K_{k_{2n-3}'} + \mathrm{i}K_{k_{2n-2}'})\\ &\qquad\times \delta_{k_1'k_1}\delta_{k_2'k_2}\ldots\delta_{k_{2n-3}'k_{2n-3}}\delta_{k_{2n-2}'k_{2n-2}}\end{aligned} \tag{57}$$

and

$$\begin{aligned}&\mathbf{X}(n,k_1k_2,k_3k_4,\ldots;n+1,k_1'k_2',k_3'k_4',\ldots)\\ &\quad\simeq \mathbf{Y}^{N-1}_{(0;0)}(N-n)\mathbf{Y}(0;k_{2n+2}';k_{2n+3}')\,\delta_{k_1'k_1}\delta_{k_2'k_2}\ldots\delta_{k_{2n}'k_{2n}}\delta_{k_{2n+2}'k_{2n+2}}\\ &\quad= \mathbf{Y}^N_{(0;0)}(N-n)D^*_{-k_{2n+2}'+k_{2n+3}'}\exp(\mathrm{i}K_{k_{2n}'} + \mathrm{i}K_{k_{2n+2}'} + \ldots)\\ &\qquad\times \delta_{k_1'k_1}\delta_{k_2'k_2}\ldots\delta_{k_{2n}'k_{2n}}\delta_{k_{2n+2}'k_{2n+2}}.\end{aligned} \tag{58}$$

The nth-order diagonal block has dimensions

$$L^{4n}/n! \tag{59}$$

where the first factor represents the number of different choices of the $2n$ independent $\boldsymbol{k}$, and the second factor recognises that any of the n pairs, $\boldsymbol{k}_1\boldsymbol{k}_2$, can be permuted among the n $\mathbf{Y}$ matrices.

The equations given above define $\mathbf{X}$, but we shall be interested in the modulus of the reflections coefficient and therefore we need to construct $\mathbf{X}\otimes\mathbf{X}^*$. In the following equations we have taken averages over fluctuations in D:

$$\begin{aligned}
&\mathbf{X}(n, \boldsymbol{k}_1\boldsymbol{k}_2, \boldsymbol{k}_3\boldsymbol{k}_4, \ldots; n, \boldsymbol{k}'_1\boldsymbol{k}'_2, \boldsymbol{k}'_3\boldsymbol{k}'_4, \ldots)\\
&\qquad\otimes\mathbf{X}^*(n, \boldsymbol{k}_{*1}\boldsymbol{k}_{*2}, \boldsymbol{k}_{*3}\boldsymbol{k}_{*4}, \ldots; n, \boldsymbol{k}'_{*1}\boldsymbol{k}'_{*2}, \boldsymbol{k}'_{*3}\boldsymbol{k}'_{*4}, \ldots)\\
&\quad=\{(1-n\delta^2)\delta_{k'_1k_1}\delta_{k'_2k_2}\delta_{k'_3k_3}\delta_{k'_4k_4}\cdots\delta_{k'_{*1}k_{*1}}\delta_{k'_{*2}k_{*2}}\delta_{k'_{*3}k_{*3}}\delta_{k'_{*4}k_{*4}}\cdots\\
&\qquad+[\delta^2/(4L^2)]\delta_{k_2-k_{*2},k'_2-k'_{*2}}\delta_{k_1k'_1}\delta_{k'_3k_3}\delta_{k'_4k_4}\cdots\\
&\qquad\times\delta_{k_{*1}k'_{*1}}\delta_{k'_{*3}k_{*3}}\delta_{k'_{*4}k_{*4}}\cdots+[\delta^2/(4L^2)]\delta_{k_1-k_{*1},k'_1-k'_{*1}}\\
&\qquad\times\delta_{k_2k'_2}\delta_{k'_3k_3}\delta_{k'_4k_4}\cdots\delta_{k_{*2}k'_{*2}}\delta_{k'_{*3}k_{*3}}\delta_{k'_{*4}k_{*4}}\cdots\\
&\qquad+[\delta^2/(4L^2)]\delta_{k_{*1}+k_2,k'_{*1}+k'_2}\delta_{k_1k'_1}\delta_{k'_3k_3}\delta_{k'_4k_4}\cdots\\
&\qquad\times\delta_{k_{*2}k'_{*2}}\delta_{k'_{*3}k_{*3}}\delta_{k'_{*4}k_{*4}}\cdots\\
&\qquad+[\delta^2/(4L^2)]\delta_{k_1+k_{*2},k'_1+k'_{*2}}\delta_{k_2k'_2}\delta_{k'_3k_3}\delta_{k'_4k_4}\delta_{k_{*1}k'_{*1}}\delta_{k'_{*3}k_{*3}}\delta_{k'_{*4}k_{*4}k_{*4}}\cdots\\
&\qquad-[n\delta^2/(4L^2)]\delta_{k'_1-k'_2,k_1-k_2}-[n\delta^2/(4L^2)]\delta_{k'_{*1}-k_{*2},k_{*1}-k_{*2}}\}\\
&\qquad\times\exp(\mathrm{i}\boldsymbol{K}_{k'_1}+\mathrm{i}\boldsymbol{K}_{k'_2}+\ldots-\mathrm{i}\boldsymbol{K}_{k'_{*1}}-\mathrm{i}\boldsymbol{K}_{k'_{*2}}-\ldots)
\end{aligned}\tag{60}$$

$$\begin{aligned}
&\mathbf{X}(n, \boldsymbol{k}_1\boldsymbol{k}_2, \boldsymbol{k}_3\boldsymbol{k}_4, \ldots; n-1, \boldsymbol{k}'_1\boldsymbol{k}'_2, \boldsymbol{k}'_3\boldsymbol{k}'_4, \ldots)\\
&\qquad\otimes\mathbf{X}^*(n, \boldsymbol{k}_{*1}\boldsymbol{k}_{*2}, \boldsymbol{k}_{*3}\boldsymbol{k}_{*4}, \ldots; n-1, \boldsymbol{k}'_{*1}\boldsymbol{k}'_{*2}, \boldsymbol{k}'_{*3}\boldsymbol{k}'_{*4}, \ldots)\\
&\quad\simeq[\delta^2/(4L^2)]\delta_{k_{2n}-k_{2n+1},k_{*2n}-k_{*2n+1}}\delta_{k_1k'_1}\delta_{k'_2k_2}\cdots\\
&\qquad\times\delta_{k'_{2n-2},k_{2n-2}}\delta_{k'_{*1}k_{*1}}\delta_{k'_{*2}k_{*2}}\cdots\delta_{k'_{*2n-2},k_{*2n-2}}\\
&\qquad\times\exp(\mathrm{i}\boldsymbol{K}_{k'_1}+\mathrm{i}\boldsymbol{K}_{k'_2}+\ldots-\mathrm{i}\boldsymbol{K}_{k'_{*1}}-\mathrm{i}\boldsymbol{K}_{k'_{*2}}-\ldots)
\end{aligned}\tag{61}$$

and

$$\begin{aligned}
&\mathbf{X}(n, \boldsymbol{k}_1\boldsymbol{k}_2, \boldsymbol{k}_3\boldsymbol{k}_4, \ldots; n+1, \boldsymbol{k}'_1\boldsymbol{k}'_2, \boldsymbol{k}'_3\boldsymbol{k}'_4, \ldots)\\
&\qquad\otimes\mathbf{X}^*(n, \boldsymbol{k}_{*1}\boldsymbol{k}_{*2}, \boldsymbol{k}_{*3}\boldsymbol{k}_{*4}, \ldots; n+1, \boldsymbol{k}'_{*1}\boldsymbol{k}'_{*2}, \boldsymbol{k}'_{*3}\boldsymbol{k}'_{*4}, \ldots)\\
&\quad\simeq[n^2\delta^2/(4L^2)]\delta_{-k'_{2n+2}+k'_{2n+3},-k'_{*2n+2}+k'_{*2n+3}}\\
&\qquad\times\delta_{k'_1k_1}\delta_{k'_2k_2}\cdots\delta_{k'_{2n+2}k_{2n+2}}\delta_{k'_{*1}k_{*1}}\delta_{k'_{*2}k'_2}\cdots\delta_{k'_{*2n+2}k_{*2n+2}}\\
&\qquad\times\exp(\mathrm{i}\boldsymbol{K}_{k'_{2n}}+\mathrm{i}\boldsymbol{K}_{k'_{2n+2}}+\ldots-\mathrm{i}\boldsymbol{K}_{k''_{2n}}-\mathrm{i}\boldsymbol{K}_{k''_{2n+2}}+\ldots).
\end{aligned}\tag{62}$$

The diagonal block (60) appears to be complex, but in reality is rather simple: the off-diagonal terms within this block allow for all possible scatterings between all possible pairs of excitations, but conserving momentum parallel to the layers. These are the terms that will describe the peaks in forward and back-scattering, the latter described by the maximally crossed diagrams (Bergman 1984). We shall not consider them further here.

5. Transport in three-dimensions—a classical model

The transfer matrix for the 3D case in the complex entity we would expect in order to describe the rich variety of phenomena that can occur. Our objective in this section is to show that in the weak-disorder limit a transfer matrix approach gives the characteristic $1/L_z$ Ohm's law behaviour. The strategy is to show which terms can be left out of the 3D matrix when the disorder is weak, and then to map the remaining terms onto the 1D classical problem, which we have already solved. We shall show that the 1D classical conductivity can be identified with the conductivity per unit area of the 3D system.

We take a minimalist approach in which only those terms which have no phase information are retained. Out justification is that we shall be concerned with the weak scattering limit in which the solutions are of a classical diffusive nature, and we expect phase information to be irrelevant. In the case of the diagonal elements this means that every one of the $2n$ values of $\boldsymbol{k}$ in $\mathbf{X}$ must be matched with a corresponding $\boldsymbol{k}$ in $\mathbf{X}^*$, which ensures that no phases appear in $\mathbf{X} \otimes \mathbf{X}^*$ and all the elements are real.

The diagonal blocks have degenerate elements on the diagonal,

$$(1 - \delta^2 n). \tag{63}$$

Each nth-order diagonal element is coupled to $(L^2)^2$ elements in the $(n+1)$th-order diagonal block via a matrix element, which after averaging is

$$\delta^2 n^2/(4L^2) \tag{64}$$

and to $(L^2)^2$ elements in the $(n-1)$th-order diagonal block via a matrix element, which after averaging is

$$\delta^2/(4L^2). \tag{65}$$

Putting these elements together in a diagram simplifies this otherwise confusing picture,

$$\overline{\mathbf{X} \otimes \mathbf{X}^*} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ \delta^2/(4L^2) & (1-\delta^2 n/2)\delta_{jj'} & \delta^2 n^2/(4L^2) & 0 \\ & + \text{off-diagonal} & & \\ & \text{coupling} & & \\ 0 & \delta^2/4L^2 & [1-\delta^2(n+1)/2]\delta_{jj'} & \delta^2(n+1)^2/(4L^2) \\ & & + \text{off-diagonal} & \\ & & \text{coupling} & \\ 0 & 0 & \delta^2/(4L^2) & \cdots \end{bmatrix}. \tag{66}$$

We must consider one further portion of this matrix: that which contains the reflection matrix. We saw in paper I that the (0, 1) block of $\mathbf{X}^N$ contains the elements

$$N\boldsymbol{r}_{-k_1'-k_2'} \tag{67}$$

and the (0, 1) of $\mathbf{X} \otimes \mathbf{X}^*$ contains

$$NN^*\boldsymbol{r}_{-k_1'-k_2'}\boldsymbol{r}^*_{-k'_{*1}-k'_{*2}}. \tag{68}$$

Furthermore, the subscripts are restricted by our selection of elements to those which occur when we take the matrix product tr $\boldsymbol{r}\boldsymbol{r}^+$. This will be important when we examine the results of the projection operation discussed below.

$\mathbf{X} \otimes \mathbf{X}^*$ can be reduced to a real symmetric matrix by a block-diagonal unitary transformation based on

$$\mathbf{Q}_{nn'} = \delta_{nn'}(n-1)!. \tag{69}$$

Since the reflection coefficient remains bounded as L_z goes to infinity, the spectrum of $\mathbf{X} \otimes \mathbf{X}^*$ cannot contain an eigenvalue of modulus larger than unity. If unphysical oscillations with L_z are to be avoided, then the largest eigenvalue must be real. Therefore we have a variational principle: if we guess a trial form for the eigenvectors of $\mathbf{X} \otimes \mathbf{X}^*$ we can use our guess to estimate the eigenvalue, and if by chance our estimate turns out to be unity, we know that we have made a good guess for the eigenvector. This result is not true of the general form of $\mathbf{X} \otimes \mathbf{X}^*$ because it is neither real symmetric nor Hermitian.

We are ready to transform $\mathbf{X} \otimes \mathbf{X}^*$ to the equivalent 1D matrix. We select from the nth-order elements those for which the $\boldsymbol{k}_1^*$ are a permutation of the $\boldsymbol{k}_1$, and the $\boldsymbol{k}_2^*$ are a permutation of the $\boldsymbol{k}_2$. Next we introduce the restriction that at least one pair $\boldsymbol{k}_1\boldsymbol{k}_2$ in $\mathbf{X}$ must be matched with a corresponding $\boldsymbol{k}_1^*\boldsymbol{k}_2^*$ pair in $\mathbf{X}^*$. The nth-order block on the diagonal has dimensions

$$L^{4n}/n$$

which comes about from the fact that we have $2n$ variables $\boldsymbol{k}$ each of which can take L^2 values. Each pair $\boldsymbol{k}_1\boldsymbol{k}_2$ can occur in any order, factor of $1/n!$, remembering that one of the pairs is fixed; $(n-1)$ of the $\boldsymbol{k}_2$ can be permuted among the pairs to generate a total of $(n-1)!$ combinations. Our trial form of eigenvector consists of reducing the nth-order blocks by a projection onto this subspace of the form

$$\boldsymbol{v}_n^{\mathrm{r}} = \{n/[n!(L^{4n})^{1/2}]\}(1, 1, 1, 1, \ldots L^{4n}/n \text{ terms})^{\mathrm{T}} \tag{70}$$

$$\boldsymbol{v}_n^{\mathrm{l}} = \{n!/L^{4n})^{1/2}\}(1, 1, 1, 1, \ldots L^{4n}/n \text{ terms}). \tag{71}$$

Within this space our matrix is now tridiagonal rather than block tridiagonal,

$$\overline{(\mathbf{X} \otimes \mathbf{X}^*)_{nn'}} = \begin{bmatrix} \ddots & \cdots & \cdots & \cdots & \cdots \\ n\delta^2/4 & 1-n\delta^2/2 & n\delta^2/4 & 0 & \cdots \\ 0 & (n+1)\delta^2/4 & 1-(n+1)\delta^2/2 & (n+1)\delta^2/4 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{72}$$

Comparing with the 1D equations we see that our objective of transforming the 3D quantum problem to the 1D classical problem has been achieved in the weak scattering limit, provided that we identify

$$r = \delta^2/4. \tag{73}$$

As we hoped, this matrix has an eigenvalue spectrum extending up to a maximum of unity.

Now let us proceed to calculate the reflected intensity, contained in the (0, 1) element of $(\mathbf{X} \otimes \mathbf{X}^*)_{nn'}$,

$$NN^* \sum_{k_1' k_2'} (\boldsymbol{r}_{L_z})_{-k_1'-k_2'} (\boldsymbol{r}_{L_z})^*_{-k_1'-k_2'}/L^2$$

$$= NN^* \operatorname{tr} \boldsymbol{r}_{L_z} \boldsymbol{r}_{L_z}^{\dagger}/L^2 \tag{74}$$

where $\mathrm{tr}(\mathbf{1} - \boldsymbol{r}_{L_z}\boldsymbol{r}_{L_z}^{\dagger})/\boldsymbol{L}^2$ is the conductance of the system per unit area and transformation relates this quantity to the classical reflectivity, r_{L_z}.

The expression for $(\mathbf{X} \otimes \mathbf{X}^*)_{nn'}$ given above refers to a single layer of the 3D solid. Putting together L_z of these layers is achieved by taking the L_z power of this matrix, which can be done using the methods of the 1D problem. In fact we are interested only in the first row because that is where the reflection matrix is held:

$$\begin{aligned}\overline{(\mathbf{X}_{L_z} \otimes \mathbf{X}_{L_z}^*)_{01}} &= [1/(L^4)^{1/2}]\,\overline{(\mathrm{tr}\,\boldsymbol{r}_{L_z}\boldsymbol{r}_{L_z}^{\dagger})} + \mathrm{O}(1/L^2) \\ &= 1 - \{1 + [(\delta^2/4)/(1 - \delta^2/4)]L_z\}^{-1} \\ &\simeq [1 - 4/(\delta^2 L_z)]. \end{aligned} \tag{75}$$

Recognising that

$$\begin{aligned}\overline{\mathrm{tr}\,\boldsymbol{r}_{L_z}\boldsymbol{r}_{L_z}^{\dagger}} &= \overline{\mathrm{tr}(\mathbf{1} - \boldsymbol{t}_{L_z}\boldsymbol{t}_{L_z}^{\dagger})} \\ &= L^2(1 - G_{L_z}) \end{aligned} \tag{76}$$

where G_{L_z} is the conductance, we obtain

$$G_{L_z} = 4/(\delta^2 L_z). \tag{77}$$

In other words, even quantum systems obey Ohm's law when the disorder is weak enough. Our 3D transfer matrix approach has correctly reproduced this result. The next challenge is to see if it can describe the localised regime.

6. Transport in three dimensions—the quantum case

In our model we have the option of switching off the hopping between the parallel chains comprising the 3D array and thus reverting to an array of independent 1D systems. We should be able to retrieve from our formalism the 1D limit, which as we know represents localised behaviour. Again our objective will be to transform the 3D equations to the 1D touchstone, in this case the equation (35).

Returning to the 3D transfer matrix equations, (60)–(62), we see that in the 1D limit there is no dispersion of $\boldsymbol{K}_k$ with $\boldsymbol{k}$. As a result all the phase factors are unity and instead of a subset of diagonal elements being degenerate, all of them are now degenerate to order δ^2 and they are all coupled together by various off-diagonal elements. Once again exploiting the separable nature of the off-diagonal elements, we implement a transformation to a new basis set defined by the following right and left vectors within the space of the nth-order block:

$$\boldsymbol{v}_n^{\mathrm{r}} = [1/(L^{8n})^{1/2}](1, 1, 1, 1, \ldots (L^{8n}/n!)^2 \text{ terms})^{\mathrm{T}} \tag{78a}$$

$$\boldsymbol{v}_n^{\mathrm{l}} = [(n!)^2/L^{8n})^{1/2}](1, 1, 1, 1, \ldots (L^{8n}/n!)^2 \text{ terms}). \tag{78b}$$

Again, because of the separable property, the matrix outside the space defined by these vectors becomes trivially diagonal and we shall ignore this space. Within the non-trivial

space our matrix is now tridiagonal rather than block tridiagonal,

$$\overline{(\mathbf{XQ}\otimes\mathbf{XQ}^*)_{nn'}} = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots & \cdots \\ n^2\delta^2/4 & 1-n^2\delta^2/2 & n^2\delta^2/4 & 0 & \cdots \\ 0 & (n+1)^2\delta^2/4 & 1-(n+1)^2\delta^2/2 & (n+1)^2\delta^2 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}. \tag{79}$$

Note that the crucial distinction between (79) and the corresponding delocalised equation (72) is the occurrence of n^2 rather than n. Comparing with the 1D equation (35) we see that our objective of transforming the 3D quantum problem to the 1D classical problem has been achieved in the limit of independent chains.

Now let us proceed to calculate the reflected intensity and its moments. The first column of $(\mathbf{XQ}\otimes\mathbf{XQ}^*)_{nn'}$ contains these quantities. As we have shown the nth block of the first column consists of terms such as

$$r_{k_1k_2}r^*_{k_1'k_2'}\ldots r_{k_{2n+1}k_{2n+2}}r^*_{k_{2n+1}'k_{2n+2}'} \tag{80}$$

Much the same as we found in equation (70) except that now all values of $k_1 \ldots k'_{2n+1}$ are present. The projection vectors lead to a sum over all the subscripts,

$$\overline{(\mathbf{XQ}\otimes\mathbf{XQ}^*)}_{n0} = [(n!)^2/L^{8n)^{1/2}}] \sum_{k_1k_1'k_2k_2'\ldots} r_{k_1k_2}r^*_{k_1'k_2'}\ldots r_{k_{2n+1}k_{2n+2}}r^*_{k_{2n+1}'k_{2n+2}'} \tag{81}$$

which simply represents the Fourier transformation of each of the $r_{k_1k_2}\ldots$ and $k^*_{k_1'k_2'}\ldots$ into a real-space representation,

$$r_{00} = \sum_{k_1k_2} r_{k_1k_2}/L^4 \tag{82}$$

$$r^*_{00} = \sum_{k_1'k_2'} r^*_{k_1'k_2'}/L^4 \tag{83}$$

being the reflection coefficient and its complex conjugate of the 1D chain located the origin in real space. Another transformation with a different phase factor would extract another statistically identical chain.

Therefore we have proved that in the limit of no interchain coupling our 3D formalism yields the proper 1D equations for the statistics of the reflection coefficients of the individual chains.

So we have shown that

$$\overline{(\mathbf{XQ}\otimes\mathbf{XQ}^*)_{n0}} = \overline{(r_{00}r^*_{00})^n}. \tag{84}$$

The expression for $(\mathbf{X}\otimes\mathbf{X}^*)_{nn'}$ given above refers to a single layer of the 3D solid. Putting together L_z of these layers is achieved by taking the L_z power of this matrix, which can be done using the methods of the 1D problem. In fact we are interested only in the first column because that is where the reflection matrix is held:

$$\overline{(\mathbf{XQ}_{L_z}\otimes\mathbf{XQ}^*_{L_z})_{n0}} = \overline{(r_{L_z00}r^*_{L_z00})^n}. \tag{85}$$

Recognising that

$$\overline{r_{L_z00}r^*_{L_z00}} = \overline{(1-t_{L_z00}t^*_{L_z00})} \tag{86}$$

$$= (1 - GQ_{L_z})$$

where GQ_{L_z} is the quantum conductance, we obtain

$$GQ_{L_z} = \{1/[2(\pi L_z \delta'^2)^{3/2}]\} \exp(-L_z \delta'^2/4) \tag{87}$$

the well known result for 1D systems. See Pendry and Kirkman (1984), where other references to this result will be found.

7. Conclusions

We have applied the symmetrised transfer matrix formalism to a 3D system. The method easily and naturally describes both the conducting and localised regimes: in the former case by neglect of any terms containing phases, and in the latter case by averaging over phases. Near the mobility edge a more complex solution to our equations will be necessary because of the known importance of the phases. In particular, in 2D, the phases are important even for weak disorder. We can in fact identify the maximally crossed diagrams, which give the first indications of the importance of the phases, but these must be the topic of a later paper.

Thus we have a powerful and general approach to the problem of transport in disordered systems. I envisage that the method can be used to answer not only subtle questions of correlations in the wavefield in the vicinity of these limits, but also the crucial range of parameters around the mobility edge.

Here is a method that has been shown to contain both localised and delocalised aspects, and gives the known solutions to limiting cases. Surely it is an ideal candidate for exploring the region in between?

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